Approximate Solution

The heat-conduction equation (7) is integrated over the volume of the solid

$$\int_{\xi}^{\infty} \frac{\partial}{\partial \zeta}\left[\zeta\left(\frac{\partial \theta}{\partial \zeta}\right)\right] d\zeta = \int_{\xi}^{\infty}\left[\frac{\partial(\theta\zeta)}{\partial \tau}\right] d\zeta \tag{14}$$

Use is made of the Leibnitz rule and Eqs (8) and (10) to put (14) into the form

$$(\alpha - \beta\dot{\xi})\xi = \xi\dot{\xi} + \frac{d}{d\tau}\int_{\xi}^{\infty}(\theta\zeta)d\zeta \tag{15}$$

The exponential temperature profile

$$\theta = \exp[-(\alpha - \beta\dot{\xi})(\zeta - \xi)] \tag{16}$$

satisfies boundary conditions (8–10) Combining Eqs (15) and (16), integrating, and simplifying results in the following second-order ordinary nonlinear differential equation for the position of the ablating surface ξ as a function of time:

$$\beta\ddot{\xi}[(\alpha - \beta\dot{\xi})\xi + 2] = (\alpha - \beta\dot{\xi})^2 \times \{(\alpha - \beta\dot{\xi})^2\xi - \dot{\xi}[(\alpha - \beta\dot{\xi})\xi + 1]\} \tag{17}$$

It is of interest to note that the steady-state ablation rate checks with results published in the literature[7, 8] For large time, it is reasonable to assume that $(\alpha - \beta\dot{\xi})\xi \rightarrow \infty$ Hence, with $\ddot{\xi} \rightarrow 0$, Eq (17) reduces to

$$\dot{\xi}_{\infty} = \alpha/(\beta + 1) \tag{18}$$

A short-time solution is readily found by expanding $\xi(\tau)$ in a Taylor's series about $\tau = 0$ and making use of initial conditions (11) and Eq (17) to obtain

$$\xi = 1 + (\alpha^4\tau^2)/[2\beta(\alpha + 2)] \tag{19}$$

The results of a numerical step-by-step integration of Eq (17) are presented in Fig 2 for $\beta = 10$ and a range of α extending from 10 to 1000 The parameter β, which depends only on the physical properties of the solid, is applicable to nylon Physical values of time for $10^{-7} < \tau < 1$ correspond to $10^{-3} < t < 10{,}000$ sec

In any particular problem, the probable applicability of the solution can be gaged by calculating the premelt temperature distribution and comparing it with the exponential function assumed in Eq (16) When the value of α is high (high heat input to low conductivity materials), the assumed temperature distribution drops sharply and good agreement has been obtained with the calculated premelt distribution At $t = 0$, Eq (16) reduces to

$$\theta = \exp[-\alpha(\zeta - 1)]$$

and for α sufficiently high, most of the heat absorbed by the solid is contained in a "thermal" layer δ, which is small compared to the initial radius of the cylindrical cavity For $\delta/a \ll 1$, the effects of the nonplanar geometry are minimized and results[8] for a semi-infinite medium,

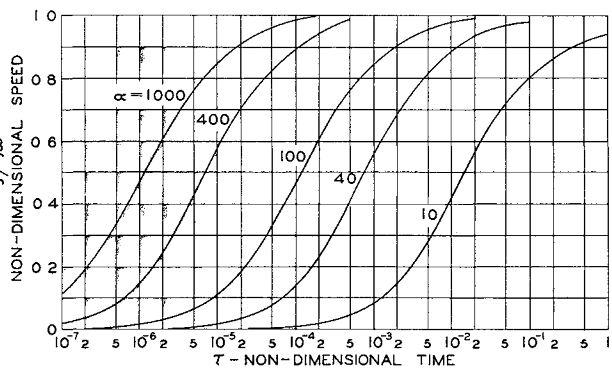

Fig 2 Ablation rate history: $\beta = 10$

ablating at its interface, will be approached in the limit For very low heat inputs $\alpha \rightarrow 0$, it is unlikely that the exponential temperature function assumed here is applicable However, many problems of practical interest will undoubtedly involve high thermal inputs

The results reported also should be applicable to finite geometries (tubes) up to the time corresponding to a rise in temperature of the unheated or outer surface

References

[1] Goodman, T R, "The heat balance integral and its application to problems involving a change of phase," Trans Am Soc Mech Engrs **80**, 335–342 (February 1958)

[2] Goodman, T R, "The heat balance integral—further considerations and refinements," Trans Am Soc Mech Engrs, J Heat Transfer **83**, 83–86 (February 1961)

[3] Goodman, T R and Shea, J J, "The melting of finite slabs," J Appl Mech **27**, 16–24 (March 1960)

[4] Goodman, T R, "The heating of slabs with arbitrary heat inputs," J Aerospace Sci **26**, 187–188 (1959)

[5] Blecher, S and Sutton, G W, "Comparison of some approximate methods for calculating re-entry ablation of a subliming material," ARS J **31**, 433–435 (1961)

[6] Lardner, T J, "Approximate solutions for melting and ablation problems," Polytechnic Institute of Brooklyn, PIBAL Rept 654 (June 1962)

[7] Baer, D and Ambrosio, A, "Heat conduction in a semi-infinite slab with sublimation at the surface," *Ballistic Missiles and Space Technology* (Pergamon Press, New York, 1961), Vol II, pp 436-446

[8] Sunderland, J E and Grosh, R J, "Transient temperature in a melting solid," Trans Am Soc Mech Engrs, J Heat Transfer **83**, 409–414 (November 1961)

Euler's Moment Equations for a Variable-Mass Unsymmetrical Top

VALDEMARS PUNGA*

Rensselaer Poltyechnic Institute of Connecticut, East Windsor Hill, Conn

Nomenclature

B	=	body of variable mass
0	=	fixed point of the body
$0x_1x_2x_3$	=	body-axes system
m	=	mass of a typical particle of the body
x_i	=	body axes coordinates of mass particle m, $i = 1,2\ 3$
ω_i	=	angular velocity of the body B expressed in body-axes system
d/dt	=	derivative with respect to any fixed (inertial) coordinate axes
$\delta/\delta t$	=	derivatives with respect to the body axes
ϵ_{ijk}	=	permutation tensor = 1, if ijk is a cyclic permutation of 1, 2, 3; −1, if ijk is an anticyclic permutation of 1, 2, 3; 0 when any two of $i, j\ k$ are equal
V_i	=	dx_i/dt = absolute velocity of particle m, i e, velocity with respect to a fixed coordinate system
c_i	=	relative velocity of mass ejected by a particle m, i e velocity relative to the body axes
u_i	=	$v_i + c_i$, absolute velocity of mass ejected by a particle m, i e, velocity relative to the fixed axes
$\dot{m}$	=	rate of the mass flow ejected by a particle m
F_i	=	external force acting on particle of mass m
f_i	=	$\dot{m}c_i$, reactive force acting on particle of mass m

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* Professor of Mathematics Member AIAA

A BODY of variable mass moves about a fixed point 0 During the motion, the body continuously ejects mass The shape of the body and, consequently, its moments and products of inertia are changing due to the mass ejection

The variable-mass body is envisaged as consisting of a collection of particles of variable mass The particles continuously eject mass, but their coordinates relative to the coordinate axes attached to the body remain invariant

Mass is ejected with a nonzero velocity relative to the body axes, and consequently reactive forces are produced Thus, a torque relative to the fixed point 0 of the body is produced by the external and reactive forces acting on the body It is assumed that the ejected mass, after its separation from the body, does not affect in any way the motion of the body

Cartesian tensors[1-3] will be used in derivations Extending the summation over all particles belonging to the body at the instant t, we further define $N_i = \Sigma\epsilon_{ijk}x_jF_k$ = torque produced by the external forces, $n_i = \Sigma\epsilon_{ijk}x_j\dot{m}c_k$ = torque produced by the reactive forces, and $H_i = \Sigma\epsilon_{ijk}x_jmV_k$ = the angular momentum of body B with respect to 0

Since $V_i = \epsilon_{ijk}\omega_jx_k$, the expression for H_i can be further modified in the following way:

$$H_i = \Sigma\epsilon_{ijk}x_jmV_k = \Sigma m(x_qx_q\delta_{ij} - x_ix_j)\omega_j = I_{ij}\omega_j \quad (1)$$

where $I_{ij} = \Sigma m(x_qx_q\delta_{ij} - x_ix_j)$, inertia tensor of the variable-mass body B

Generally, during the mass ejection, the inertia tensor may change Therefore, differentiating the angular momentum with respect to the body axes,

$$\frac{\delta H_i}{\delta t} = \frac{\delta I_{ij}}{\delta t}\omega_j + I_{ij}\frac{\delta\omega_j}{\delta t}, \qquad \frac{\delta I_{ij}}{\delta t} \neq 0 \quad (2)$$

On the other hand, differentiating $H_i = \Sigma\epsilon_{ijk}x_jmV_k$ with respect to the fixed coordinate system, we obtain

$$dH_i/dt = \Sigma\epsilon_{ijk}x_j(d/dt)(mV_k) \quad (3)$$

These are related by the standard equation $dH_i/dt = \delta H_i/\delta t + \epsilon_{ijk}\omega_jH_k$, giving

$$(\delta H_i/\delta t) + \epsilon_{ijk}\omega_jH_k = \Sigma\epsilon_{ijk}x_j(d/dt)(mV_k) \quad (4)$$

In order to express the right member of (4) in a new form, we consider the equation of motion for a single particle of variable mass m as[4]

$$m(du_i/dt) = F_i + \dot{m}c_i \quad (5)$$

Adding $\dot{m}v_i$ to both sides of (5), we obtain

$$(d/dt)(mV_i) = F_i + \dot{m}u_i \quad (6)$$

Forming a cross product of x_i and (6), and summing over all particles of the system, we obtain

$$\Sigma\epsilon_{ijk}x_j(d/dt)(mV_k) = \Sigma\epsilon_{ijk}x_jF_k + \Sigma\epsilon_{ijk}x_j\dot{m}u_k \quad (7)$$

Substituting $u_i = c_i + V_i$ and $V_i = \epsilon_{ijk}\omega_jx_k$ into the right member of (7), we obtain

$$\Sigma\epsilon_{ijk}x_j(d/dt)(mV_k) = N_i + n_i + \Sigma\epsilon_{ijk}x_j\dot{m}(\epsilon_{kpq}w_px_q) \quad (8)$$

The last term of (8) can be further expanded into the form

$$\Sigma\epsilon_{ijk}x_j\dot{m}\epsilon_{kpq}\omega_px_q = \Sigma\dot{m}(x_qx_q\delta_{ij} - x_ix_j)\omega_j \quad (9)$$

Now $\dot{m} = dm/dt = \delta m/\delta t$, since the rate of mass ejection is the same when measured by observers stationed in fixed or moving coordinate systems Equation (9) can be modified to the form

$$\Sigma\epsilon_{ijk}x_j\dot{m}\epsilon_{kpq}\omega_px_q = \Sigma(\delta m/\delta t)(x_qx_q\delta_{ij} - x_ix_j)\omega_j = \omega_j(\delta/\delta t)[m(x_qx_q\delta_{ij} - x_ix_j)] = (\delta I_{ij}/\delta t)\omega_j \quad (10)$$

since coordinates x_i of particles m are constants with respect to the body axes Combining Eqs (1, 2, 4, 8, and 10), we obtain Euler's dynamical equation for the body of variable mass moving about a fixed point:

$$I_{ij}(\delta\omega_j/\delta t) + \epsilon_{ijk}\omega_jI_{kq}\omega_q = N_i + n_i \quad (11)$$

An interesting feature of Eq (11) is the fact that $\delta I_{ij}/\delta t$ does not appear

References

[1] Jeffreys, H , *Cartesian Tensors* (Cambridge University Press, Cambridge, England, 1957)

[2] Duschek, A and Hochrainer, A *Grundzüge der Tensorrechnung in Analytischer Darstellung* (Springer-Verlag, Wien, 1950)

[3] Narayan, S , *A Text Book of Cartesian Tensors* (S Chand and Co , Delhi, India, 1956)

[4] Miele, A , *Flight Mechanics—I* (Addison-Wesley Publishing Company, Inc , Reading, Mass , 1962), p 24

Extension of Lifting-Line Theory to a Cascade of Split Aerofoils

B LAKSHMINARAYANA*

Pennsylvania State University, University Park, Pa

Nomenclature

ω	= potential function
Γ	= circulation
S	= spacing of the aerofoils
y,z	= Cartesian coordinates
u	= induced velocity at the lifting line
$2l$	= total length of the aerofoil
2τ	= gap width
B_1,B_2 B_n	= Fourier coefficients
V	= inlet velocity
$\Omega = \tau/l$	= gap/length ratio
a_0	= slope of the lift angle-of-attack curve
C	= chord length
β	= angle of attack measured from the attitude of no lift
L	= total lift
D_i	= induced drag
C_{Di}	= induced drag coefficient
ρ	= fluid density
μ	= $a_0C/8l$

ANALYTICAL and experimental investigation on the effect of a gap in a cascade aerofoil is investigated in this note Using lifting-line theory, expressions are given for induced velocity and drag of such an aerofoil The type of analysis presented has its application in predicting the tip clearance losses in axial flow turbomachines [1]

Analysis

When a gap is cut in a cascade aerofoil, the bound vortices cannot cross the gap and must, therefore, be shed off as trailing vortices Figure 1 shows the trailing vortices, and also the trailing vortex system for a cascade of split aerofoils

The potential function of a row of vortices that are infinite in one direction along the x axis (Fig 1b) is given by Milne-Thompson[2] as

$$\omega = (i\Gamma_{\text{shed}}/4S)\log\sin[\pi(y + iz)/S] \quad (1)$$

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* Visiting Assistant Professor of Aeronautical Engineering, Department of Aeronautical Engineering Member AIAA